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Crossover between short- and long-range interactions in the one-dimensional Ising model

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Abstract. We consider the one-dimensional Ising model which besides the interaction of nearest neighbours includes a long-range term. The latter is ferromagnetic while the nearest-neighbour interaction can have any sign. The crossover between short- and long-range interactions when varying the Hamiltonian parameters is analysed for such a model. For some values of the antiferromagnetic short-range interaction the transition from a ferro- to paramagnetic state is of first order. At the tricritical point where the transition order changes from second to first we obtain classical tricritical exponents. For a ferromagnetic short-range interaction the Curie temperature surpasses the transition temperature of the long-range model.

There exists an interesting question in the theory of magnetism concerning how the properties of a system change during a continuously varying interaction between two qualitatively different limiting cases. Stinchcombe [1, 2], for example, has considered the crossover between the isotropic Heisenberg and anisotropic Ising behaviours of pure and diluted magnets. The crossover with respect to the anisotropy parameter for the XY model in a transverse field has been studied by Ray and Chakrabarti [3]. Boucher *et al* [4], involving experimental data for quasi-one-dimensional antiferromagnets, have analysed the crossover between the isotropic Heisenberg and the anisotropic XY behaviour when varying an external magnetic field. Stinchcombe [5] has considered the crossover between the one- and two-dimensional behaviours when switching on an interaction between Ising chains and also the crossover from two-dimensional to three-dimensional properties for layer Heisenberg systems.

A dimensional crossover, when the space dimensionality continuously changes between three and four, has been considered by Rudnick and Nelson [6] for the Landau-Ginzburg-Wilson model and also by Rijpkema [7] for the Blume-Capel model.

Joyce [8] has obtained quite interesting results for the Berlin-Kac spherical model with a ferromagnetic interaction of the type $\mathscr{F}_{ij} \sim 1/r_{ij}^{d+\lambda}$, where d is the lattice dimensionality. As has been found, the Curie temperature and critical indices continuously depend on the parameter $\lambda > 0$. Analogous results have been obtained by Sak [9] for ferromagnets with the Joyce potential and $\lambda < 2$.

Crossovers corresponding to a change of Hamiltonian parameters or the space dimensionality should be distinguished from the crossover behaviour of the same model at a varying temperature. In the latter case one usually analyses the dependence of effective critical indices on temperature [6, 10, 11]. An introduction of effective critical indices is justified by the fact that they are directly observed in experiments [12].

In the present paper we investigate a crossover of the type where Hamiltonian parameters are changing so that it undergoes a continuous transformation between the short- and long-range forms. A combination of short- and long-range potentials is quite realistic for some materials. Magnetic alloys in which there exists a spin-orbital interaction, besides the spin-spin interaction, may serve as an example [13]. The appearance of magnetic links with different interaction radii can be also due to the presence in a magnet of defects or boundaries [14].

Let us consider the one-dimensional Ising model with the Hamiltonian

$$H = -\sum_{i,j=1}^{N} \phi_{ij} s_i s_j - B \sum_{i=1}^{N} s_i \qquad s_i = \pm 1 \tag{1}$$

in which the exchange integral is divided into two terms:

$$\phi_{ij} = I_{ij} + \mathscr{F}_{ij} \tag{2}$$

where one term corresponds to the interaction of nearest neighbours

$$I_{ij} = \begin{cases} I & i, j \text{ adjacent} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and the other term is the so-called long-range interaction which Lebowitz and Penrose [15] named the Kac potential, satisfying the properties

$$\lim_{N \to \infty} \mathscr{F}_{ij} = 0 \qquad \lim_{N \to \infty} \frac{1}{N} \sum_{i,j=1}^{N} \mathscr{F}_{ij} = \mathscr{F} < \infty. \tag{4}$$

In (3) and (4)

$$I, \mathscr{F} \in (-\infty, +\infty). \tag{5}$$

The constant I describes an input of the short-range interaction. When $\mathscr{F}_{ij} = 0$, only nearest neighbours are interacting; when $I = 0$, the system is purely of the long-range-interaction type; for $I \neq 0$, $\mathscr{F}_{ij} \neq 0$ an intermediate variant is realised. We shall consider here the case $I < 0$ as well, when the long-range interaction is ferromagnetic and the short-range one is antiferromagnetic.

Several similar models have previously been considered. Thus, Baker [16] introduced an anisotropic Ising model with an infinitely long-range force in one direction and nearest-neighbour interactions in the other directions. He examined a simple quadratic lattice and a simple cubic lattice for just one case, when the intensities of the short-range and long-range interactions have equal signs and strengths. Baker gave an approximate solution expanding the partition function about the Weiss field and certifying the phase transition of the familiar Bragg–Williams type. Suzuki [17] mentioned that a linear Ising model of the Baker kind could be solved exactly, but he did not analyse consequences of this. Nagle [18] analysed the existence of critical points for an Ising chain with positive short-range and negative long-range potentials. Later Nagle and Bonner [19] continued the consideration of such an Ising chain with competing interactions and with a term representing a staggered field acting in opposite directions on even and odd lattice sites. Theumann and Høye [20] discussed the one-dimensional Ising model with a very long-range ferromagnetic interaction and first- and second-neighbour antiferromagnetic interactions. Høye [21] also studied the linear Ising model with nearest-neighbour repulsion and infinitely long-range attraction acting only on even-numbered sites. Lapushkin and Plechko [22] treated the Ising chain with a positive long-range interaction when there is no phase transition.

Substituting (2) into (1) we obtain the Hamiltonian

$$H = -I \sum_{\langle ij \rangle} s_i s_j - \sum_{ij} \mathscr{F}_{ij} s_i s_j - B \sum_i s_i. \tag{6}$$

For $I < 0$, $\mathscr{F}_{ij} > 0$ we return to the Nagle situation [18], while for $I > 0$, $\mathscr{F}_{ij} < 0$ to the Lapushkin-Plechko case [22].

We calculate the dimensionless thermodynamic potential

$$y = -\lim_{N \to \infty} \frac{1}{N} \ln \operatorname{Tr} \exp(-H/\Theta) = f/\Theta$$

in which Θ is the temperature, f is the free energy and the Boltzmann constant $k_B \equiv 1$. We take into account that in the thermodynamic limit the long-range part of the Hamiltonian becomes equivalent to the mean-field form. More correctly, the following rigorous equality [15, 23-25] is valid:

$$y = -\lim_{N \to \infty} \frac{1}{N} \ln \operatorname{Tr}(-H_{\text{app}}/\Theta) \tag{7}$$

where the approximating Hamiltonian

$$H_{\text{app}} = -I \sum_{\langle ij \rangle} s_i s_j - 2\mathscr{F}\sigma \sum_i s_i + N\mathscr{F}\sigma^2 - B \sum_i s_i \tag{8}$$

contains the average spin

$$\sigma \equiv \langle s_i \rangle_{\text{app}} = \frac{\operatorname{Tr} s_i \exp(-H_{\text{app}}/\Theta)}{\operatorname{Tr} \exp(-H_{\text{app}}/\Theta)} \tag{9}$$

and where the translational invariance has been taken into consideration. The thermodynamic potential (7) is valid for arbitrary space dimensionalities. In what follows we deal only with the one-dimensional case. Then, using the transfer-matrix technique, we obtain

$$y = -\frac{g - \sigma^2}{T} - \ln\left\{\cosh\left(\frac{2\sigma + h}{T}\right) + \left[\sinh^2\left(\frac{2\sigma + h}{T}\right) + \exp\left(-\frac{4g}{T}\right)\right]^{1/2}\right\} \tag{10}$$

where

$$g \equiv I/\mathscr{F} \qquad T \equiv \Theta/\mathscr{F} \qquad h \equiv B/\mathscr{F}. \tag{11}$$

For the order parameter (9) we obtain

$$\sigma = \left[\sinh^2\left(\frac{2\sigma + h}{T}\right) + \exp\left(-\frac{4g}{T}\right)\right]^{-1/2} \sinh \frac{2\sigma + h}{T}. \tag{12}$$

The critical temperature satisfying the conditions $\sigma_c = 0$, $h_c = 0$, is defined by the equation

$$T_c = 2 \exp(2g/T_c) \tag{13}$$

which may be written as

$$\Theta_c = 2\mathscr{F} \exp(2I/\Theta_c).$$

In the short-range limit the critical temperature is

$$\Theta_c = 0 \qquad \mathscr{F} = 0$$

as it should be. In the mean-field case it is

$$\Theta_c = 2\mathscr{F} \qquad I = 0.$$

From equation (13) it follows that non-trivial solutions for the critical temperature exist if

$$T_c > 2/e \qquad g > -1/e \tag{14}$$

which yields

$$T_c + 2g > 0.$$

Therefore

$$\frac{\partial T_c}{\partial g} = \frac{2T_c}{2g + T_c} > 0$$

and thus the critical temperature is a monotonic function of g. Under $g \gg 1$ this temperature can greatly surpass the mean-field transition temperature

$$T_c \gg 2 \qquad g \gg 1.$$

Under negative values of I the ferro-paramagnetic transition can become of first order. The change of transition order takes place at $T = T_t$ defined by the equation

$$\begin{aligned} g_t &= -\tfrac{1}{4} \ln 3T_t \\ T_t &= 2\exp(2g_t/T_t). \end{aligned} \tag{15}$$

Such a point separating the first-order line from the critical line has been called the tricritical point by Griffiths [26, 27]. The appearance of this point is due to the presence in the Hamiltonian of competing interactions having different signs. There are other examples when tricritical points occur, again because of the competition of two-sign potentials. Thus, Krinsky and Furman [28] have considered a spin-1 Ising model containing a biquadratic exchange, a non-symmetric triple exchange and a one-size anisotropy as well as a standard exchange interaction. Tricriticality has been found and analysed by Sarbach and Fisher [29] for a many-component system with a Hamiltonian including, in addition to the usual exchange and external field, one-size anisotropies of second, fourth and sixth orders in powers of spin, and a cubic field term. An excellent review on the theory of tricritical points has been given by Lawrie and Sarbach [30].

In our case (15) gives for the tricritical point

$$T_t = 2/\sqrt{3} \qquad g_t = -\ln 3/2\sqrt{3}. \tag{16}$$

At the tricritical temperature the critical exponents, as is known [30], have jumps. We can check this by calculating the asymptotic behaviour of thermodynamic characteristics for

$$\tau \equiv (\Theta - \Theta_c)/\Theta_c = (T - T_c)/T_c \to -0.$$

The behaviour of the specific heat is given by

$$C_v \sim \begin{cases} (-\tau)^{1/2} & g \neq g_t \\ (-\tau)^{1/4} & g = g_t. \end{cases} \tag{17}$$

For the order parameter (12) we find

$$\sigma \simeq \begin{cases} A_\sigma^{1/2}(-\tau)^{1/2} & g \neq g_t \\ 1.107(-\tau)^{1/4} & g = g_t \end{cases} \tag{18}$$

where the critical amplitude is given by

$$A_\sigma = 6(T_c + 2g)/(3T_c^2 - 4).$$

For the susceptibility we obtain

$$\chi \simeq A_\chi(-\tau)^{-1} \qquad \forall g \tag{19}$$

where

$$A_\chi = T_c/4(T_c + 2g).$$

Therefore at the usual critical point the critical exponents α, β and γ for the specific heat $C_v \sim (-\tau)^{-\alpha}$, the order parameter $\sigma \sim (-\tau)^\beta$, and the susceptibility $\chi \sim (-\tau)^{-\gamma}$ have their classical values:

$$\alpha = 0 \qquad \beta = \tfrac{1}{2} \qquad \gamma = 1.$$

At the tricritical point (16) these indices change by jumping to the classical tricritical exponents:

$$\alpha_t = \tfrac{1}{2} \qquad \beta_t = \tfrac{1}{4} \qquad \gamma_t = 1. \tag{20}$$

The Rushbrooke inequality always holds as the equality

$$\alpha + 2\beta + \gamma = 2 = \alpha_t + 2\beta_t + \gamma_t.$$

The analysis given above for zero external field shows that there are two limiting values of the parameter g:

$$g_0 = -1/\mathrm{e} = -0.368$$

and

$$g_t = -\ln 3/2\sqrt{3} = -0.317.$$

When $g \leqslant g_0$, there is no ferromagnetism in the system. In the region $g_0 < g < g_t$, the ferro-paramagnetic transition is of first order. The point $g = g_t$ is the tricritical one. If $g > g_t$ the usual second-order transition occurs.

The critical behaviour of the model considered may also be investigated by means of the Landau expansion for the thermodynamic potential (10):

$$y = y(g, T, \sigma) = y(g, T, 0) + a(g, T)\sigma^2 + b(g, T)\sigma^4 + c(g, T)\sigma^6 + O(\sigma^8)$$

in which

$$a(g, T) = \frac{1}{4}\left(\frac{2}{T}\right)^2\left[T - 2\exp\left(\frac{2g}{T}\right)\right]$$

$$b(g, T) = \frac{1}{4!}\left(\frac{2}{T}\right)^4 \exp\left(\frac{2g}{T}\right)\left[3\exp\left(\frac{4g}{T}\right) - 1\right]$$

$$c(g, T) = \frac{1}{6!}\left(\frac{2}{T}\right)^6 \exp\left(\frac{2g}{T}\right)\left[-1 + 30\exp\left(\frac{4g}{T}\right) - 45\exp\left(\frac{8g}{T}\right)\right].$$

A line of critical points $T_c = T_c(g)$ determined by

$$a(g, T_c) = 0 \qquad b(g, T_c) > 0$$

is given by (13). This line ends when

$$a(g, T_t) = 0 = b(g, T_t)$$

which leads the tricritical point (16).

Although the model considered is quite simple, it possesses non-trivial properties because of the competition between short- and long-range interactions.

Moreover, even this simple model can find practical applications when interpreting experimental data for quasi-one-dimensional magnets or for treating some other one-dimensional systems, such as molecular chains which can be described by Ising-type models.

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